Markov's Theorem

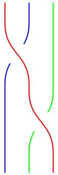
Group 3

July 27, 2020

Outline

Uh, what's a braid again?

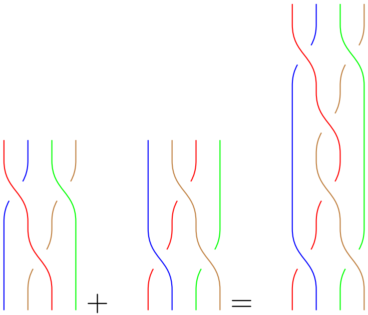
Geometrically, a braid is a group of "strings" which pass from the top of the unit square to the bottom in a monotonic manner, and which permute the set $\{1, \ldots, n\}$ in some manner for some $n \in \mathbb{N}$.



This is a braid on three strings

Algebraically describing a braid

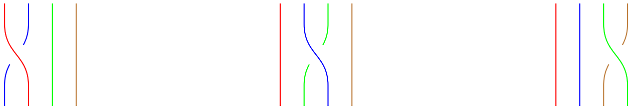
As you might remember, the set of braids on n strings, called B_n, forms a group, where the group operation is composition of braids.



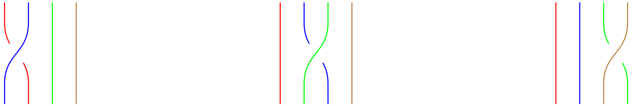
Addition of Braids

Group representation

The group B_n has the group presentation $\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i$ when $|i-j| > 2|$, $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\rangle$. Here, σ_i refers to the braid in which every strand is straight except the ith strand crosses over the $i+1$th strand. Here are the generators of B_4 as an example:



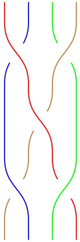
σ_1 σ_2 σ_3



σ_1^{-1} σ_2^{-1} σ_3^{-1}

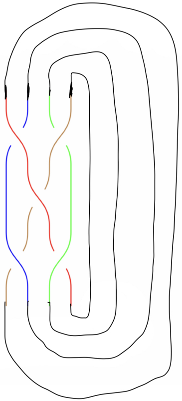
Alexander's Theorem

We can now discuss the notion of the closure of a braid. The closure of a braid is the link obtained by drawing an arc connecting the ith spot on the bottom with the ith spot on the top, as follows.



The Braid
$\sigma_1\sigma_3^{-1}\sigma_2\sigma_3^{-1}\sigma_1$

$\rightarrow$



Its closure

Alexander's Theorem (continued)

A few weeks ago, the following theorem was proven:

Important Theorem of Alexander

Any link is the closure of some braid

Immediate Observation

The braids $\sigma_1, \ldots, \sigma_i$ all have isotopic closures

Natural Follow-Up Question

When do two braids have isotopic closures? Is there a necessary AND sufficient way to characterize all such equivalence classes of braids?

Markov's Theorem

Markov's theorem characterizes braids which have isotopic closures using so-called "Markhov Moves," which transform one braid into another (not necessarily with the same number of strands). Before we get into that though, we shall take the following theorem for granted.

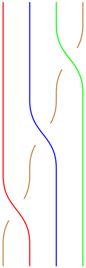
Black Box Theorem

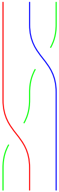
Two braids $\beta, \beta' \in B_n$ are conjugate, meaning there exists some $\gamma \in B_n$ such that $\gamma^{-1}\beta\gamma = \beta'$ if and only if they have closures which are isotopic in the solid torus V.

At first glance, this would seem to solve the issue. However, there are some wrinkles.

Wrinkle 1

Consider the following braids:





Exercise (easy)

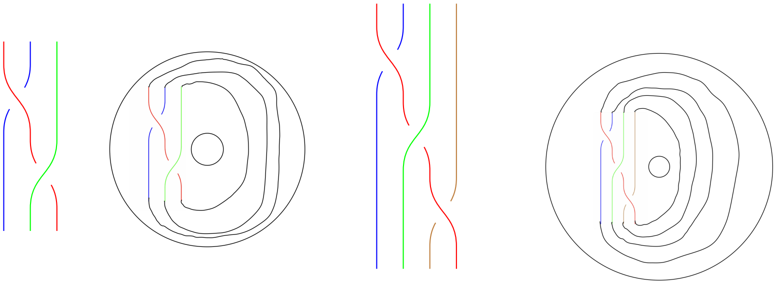
Show that these braids have closure which are isotopic in $\mathbb{R}^3$

Exercise (very easy)

Show that these braids are not conjugate in B_3

Wrinkle 2

The theorem specifies that the closures are isotopic in the solid torus V. Consider the following braids and their closures in the torus.



Exercise (easy)

Show the closures of these two braids are isotopic in $\mathbb{R}^3$

Exercise

Show the closures of these two braids are not isotopic in V

Markov Moves

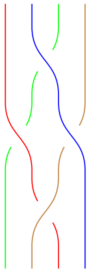
As we can see, conjugacy is not quite enough. I will now introduce 2 "moves" (and their inverses) which can be performed on a braid and produces an isotopic closure.

M1

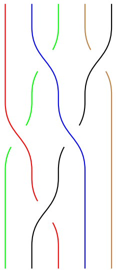
For $\beta, \gamma \in B_n$, the transformation $\beta \to \gamma\beta\gamma^{-1}$ is called the first Markov Move, and is denoted $M1$. As we can see, this is simply the conjugation of β by γ. We can see by the first black box that if two braids are $M1$ equivalent, then they have isotopic closures.

M2

For the second Markov move, we define $M2$ as the transformation $B_n \ni \beta \to \sigma_{n+1}^{\epsilon}\iota(\beta) \in B_{n+1}$, where $\epsilon = \pm 1$ and $\iota : B_n \to B_{n+1}$ is the inclusion mapping (basically adding another string on the right).



Before $M2$



After $M2$

Exercise (easy)

Show that these two braids have isotopic closures.

M-equivalence

If two braids $\beta \in B_i, \beta' \in B_j$, (with i and j not necessarily the same), are M-equivalent, this means that β can be obtained from β' by a finite sequence of moves $M1$, $M2$, $M1^{-1}$ and $M2^{-1}$. We denote this $\beta \sim \beta'$.

Exercise (easy)

Show that a move of type $M1^{-1}$ is also of type $M1$. Is this the same for $M2^{-1}$ and $M2$? (Obviously not!)

Exercise (easy)

Show that $\sim$ is an equivalence relation over $\dot{\bigcup}_{\mathbb{N}} B_n = \mathbb{B}$, the disjoint union of all braid groups on finitely many strings.

Exercise (less easy)

Show that $\sigma_1 \sim \sigma_1^{-1} \in B_2$ without using Markov's Theorem.

Markov's Theorem

Very Important Theorem of Markov

Two braids (possibly with different numbers of strings) have isotopic closures in Euclidean space $\mathbb{R}^3$ if and only if these braids are M-equivalent.

For the rest of this presentation, I will be giving you the broad strokes of the proof of Markov. It will proceed in the following fashion.

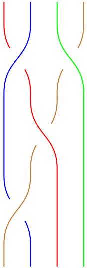
Sketch of proof

1. First, we will introduce a third Markov Move, $M3$, which is a composition of $M1$ and $M2$.
2. Next, we will use the first black box to reformulate Markov's theorem in terms of closed braids in the solid torus V, and hence reduce the proof of Markov's theorem to the proof of Lemma 1.
3. Then, we will reformulate Lemma 1 in terms of closed braid diagrams in an annulus, and hence to prove Lemma 1 it will suffice to prove another lemma, Lemma 2.
4. We shall then reduce Lemma 2 to a claim formulated in terms of so-called 0-diagrams in $\mathbb{R}^2$ representing isotopic oriented links in $\mathbb{R}^3$, hence reducing Lemma 2 to Lemma 3.
5. We shall then reduce Lemma 3 to Lemmas 4 and 5.
6. We shall then sketch a proof of Lemma 4.
7. At the end, we will decide that enough is enough and take Lemma 5 for granted, hence proving the theorem.

Let us begin.

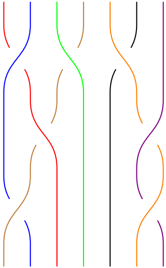
M3

If $\beta \in B_n$ and $\alpha \in B_m$, then $\beta \otimes \alpha \in B_{n+m}$ is defined as the braid consisting of β and α next to each other with no mutual crossings. Here is an example:



$\beta \in B_4$

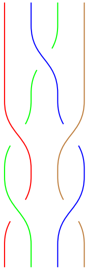
$\alpha \in B_3$



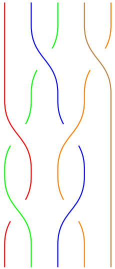
$\beta \otimes \alpha \in B_7$

$M3$ (continued)

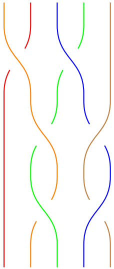
We see that by definition, $M2$ transforms a braid $\beta \in B_n$ into $\sigma_n^\epsilon(\beta \otimes 1_1)$, with $\epsilon = \pm 1$. Define $M3$ as transforming $\beta \in B_n$ into $\sigma_1^\epsilon(1_1 \otimes \beta)$. Once again, here is an example for clarity:



$\beta \in B_4$



After $M2$



After $M3$

M3 (continued) (continued)

It is the case that $M3$ expands as a composition of the moves $M1$ and $M2$, and hence $M1, M2, M3$ generate the same equivalence relation

Exercise (pretty dang hard)

Let $\Delta_n = (\sigma_1\sigma_2 \dots \sigma_{n-1})(\sigma_1\sigma_2 \dots \sigma_{n-2}) \dots (\sigma_1\sigma_2)\sigma_1 \in B_n$ (this can be obtained from the trivial braid 1_n by a half-twist achieved by keeping the top of the braid fixed and turning over the row of the lower ends by an angle of π). Using the fact that, for all $n \geq 1$ and $i = 1, \dots, n-1$, we have $\Delta_n \sigma_i \Delta_n^{-1} = \sigma_{n-i} \in B_n$, deduce that the above assertion is true. Hint: The punchline is that

$$\sigma_1^{\epsilon}(1_1 \otimes \beta) = \Delta_{n+1}^{-1}\sigma_n^{\epsilon}(\Delta_n \beta \Delta_n^{-1} \otimes 1_1)\Delta_{n+1}$$

Reduction to Lemma 1

We will reformulate Markov's theorem in terms of closed braids in the solid torus $V \subset \mathbb{R}^3$. We denote the closure of a braid β as $\hat{\beta}$. Let $\hat{M}2$ be a transformation on the set of closed braids in V which replaces $\hat{\beta}$ with $\sigma_n^{\epsilon}(\beta \hat{\otimes} 1_1)$, which is the closure of β with an $M2$ move applied to it. Define $\hat{M}3$ similarly. By the first black box, we only need to justify the following assertion to prove Markov's Theorem:

Lemma 1

Two closed braids in V representing isotopic oriented links in $\mathbb{R}^3$ can be related by a sequence of moves $\hat{M}2^{\pm 1}$, $\hat{M}3^{\pm 1}$, and isotopies in V.

We have essentially used the black box to replace $M1$ with isotopies in V.

Reformulation of Lemma 1

We will now reformulate Lemma 1 in terms of closed braid diagrams in the annulus $\tilde{V} = S_1 \times [0,1] \subset \mathbb{R}^2$. Define $\tilde{M}2$ as a transformation which replaces the diagram $\tilde{\beta} \subset \tilde{V}$, which is the diagram of $\hat{\beta}$, with $\sigma_n^{\epsilon}(\beta \tilde{\otimes} 1_1)$, which is the diagram of $\hat{\beta}$ after an $\hat{M}2$ move. $\tilde{M}3$ is defined similarly. The reason we want to think about the diagrams of braid closures instead of the braid closures themselves (a mildly subtle distinction), is to introduce what are called the "braidlike Reidemeister moves" Ω_2^{br}, which is essentially the second Reidemeister move but when the orientations of both strands coincide and Ω_3^{br}, which is essentially the third Reidemeister move but when the orientations of all three strands coincide.

Reformulation of Lemma 1 (continued)

Lemma 2

Two closed braid diagrams in an annulus $\tilde{V} \subset \mathbb{R}^2$ representing isotopic oriented links in $\mathbb{R}^3$ can be related by a sequence of moves $(\Omega_2^{br})^{\pm 1}, (\Omega_3^{br})^{\pm 1}, \tilde{M}2^{\pm 1}, \tilde{M}3^{\pm 1}$, and isotopies in the class of oriented link diagrams in $\tilde{V}$.

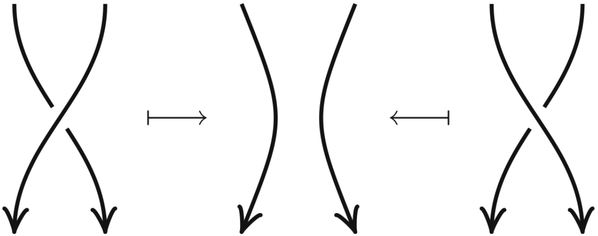
Now, what is a 0-diagram?

You may remember Vlad's talk near the beginning of Knots and Graphs where he defined smoothings and Seifert Circles. If you don't remember, here is a short review:

Definition

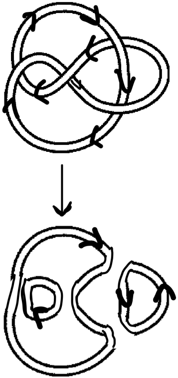
Every crossing on a knot diagram looks locally like the braid σ_1 or σ_1^{-1}. A smoothing replaces it with 1_2, the trivial braid in B_2. After performing a smoothing at every crossing of D, we end up with a set of non-intersecting circles. These are called the Seifert Circles of D. Two concentric Seifert Circles are called compatible if their orientations do not coincide, and two non-concentric Seifert Circles are called compatible if their orientations do coincide. Otherwise, they are called incompatible. The height, $h(D)$ of a knot diagram is the number of compatible pairs of Seifert Circles it has.

Examples 1

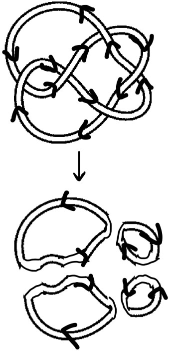


This is what a smoothing looks like

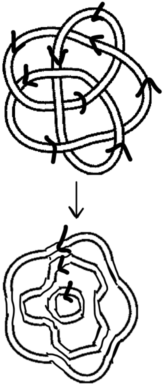
Examples 2



$h(D) = 0$. However, this is not a 0-diagram



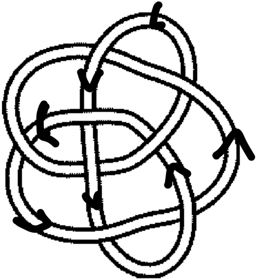
$h(D) = 2$



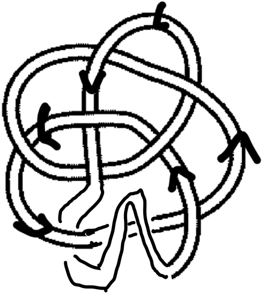
$h(D) = 0$. This is in fact a 0-diagram

Examples 3

Note that Ω_2^{br} and Ω_3^{br} take 0-diagrams to 0-diagrams

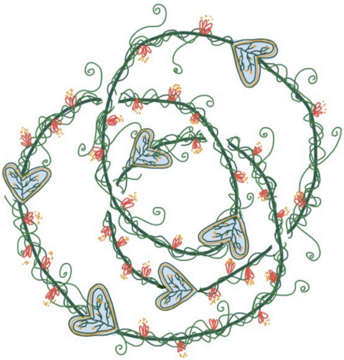


D

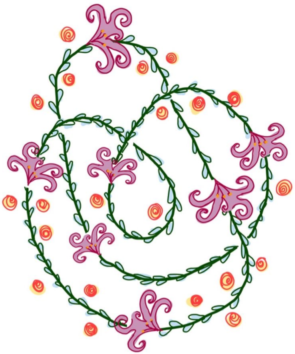


D after an application of Ω_2^{br}

Examples 4 (Illustrations courtesy of Hannah Johnson)



D

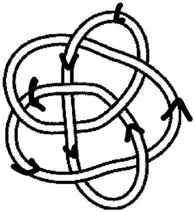


D after an application of Ω_3^{br}

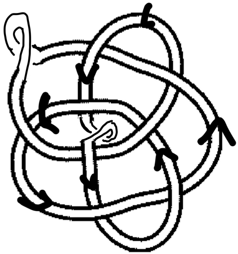
Exercise (easy)

Reidemeister 1

Usually the first Reidemeister move does not take 0-diagrams to 0-diagrams. However, there are two special cases in which it does, which we call Ω_1^{int} and Ω_1^{ext}. In the former, a loop is added on the interior of the innermost Seifert Circle, and in the latter a loop is added on the outermost Seifert Circle.



D



D after an application of Ω_1^{int} and Ω_1^{ext}

Reidemeister 1 (continued)

Exercise (easy)

Show that Ω_1^{int} and Ω_1^{ext} take 0-diagrams to 0-diagrams. For Ω_1^{ext} you can let the diagram be on $S^2 = \mathbb{R}^2 \cup \{\infty\}$.

You've probably noticed that $\tilde{M}2 = \Omega_1^{int}$ and $\tilde{M}3 = \Omega_1^{ext}$

What was the point of literally any of this?

I'm glad you asked.

Exercise (easy)

Convince yourself that every braid diagram is a 0-diagram.

We will now use the catch-all term Ω move to mean one of the following:

1. ${\Omega_2^{br}}^{\pm 1}$
2. ${\Omega_3^{br}}^{\pm 1}$
3. ${\Omega_1^{int}}^{\pm 1}$
4. ${\Omega_1^{ext}}^{\pm 1}$
5. An isotopy in $\mathbb{R}^2$

Lemma 3

Lemma 3

Two 0-diagrams in $\mathbb{R}^2$ representing isotopic oriented links in $\mathbb{R}^3$ can be related by a sequence of Ω-moves.

Lemma 3 implies Lemma 2 (which implies Lemma 1 which implies Markov's Theorem when combined with the black box).

Exercise (easy)

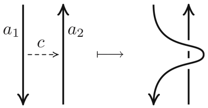
Why?

Lemmas 4 and 5

We're almost there, so don't despair! We know that Lemma 3 starts a sort of chain reaction which ends in Markov's Theorem. Lemma 4 further reduces the proof of Markov's theorem to a question about bendings, which you also may remember from Vlad's presentation.

Definition

Suppose you have a knot diagram D, and suppose you have two edges which border the same face of D and are part of different Siefert Circles. Then a bending is taking a subarc of one edge and pushing it over the other using the second Reidemeister move.



This is a bending

Lemma 4

Lemma 4

Let $\mathcal{C}, \mathcal{C}'$ be 0-diagrams in $\mathbb{R}^2$ representing isotopic oriented links in $\mathbb{R}^3$. Then there is a sequence of 0-diagrams $\mathcal{C} = \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m = \mathcal{C}'$ such that for all $i = 1, 2, \ldots, m-1$, the diagram $\mathcal{C}_{i+1}$ is obtained from $\mathcal{C}_i$ by an Ω-move, or by a sequence of bendings, tightenings, and isotopies in the sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$.

Proof Sketch of Lemma 4

Because $\mathcal{C}$ and $\mathcal{C}'$ represent isotopic links, they can be related by a finite sequence of transformations consisting of the following oriented Reidemeister moves:

(a) $\Omega_1^{\pm 1}$
(b) $(\Omega_2^{br})^{\pm 1}, (\Omega_3^{br})^{\pm 1}$, or isotopies in $\mathbb{R}^2$
(c) nonbraidlike moves $\Omega_2^{\pm 1}$

Moves of type (b)

Let g be a move of type (b) in the sequence which is applied to a link diagram D with positive height.

Exercise

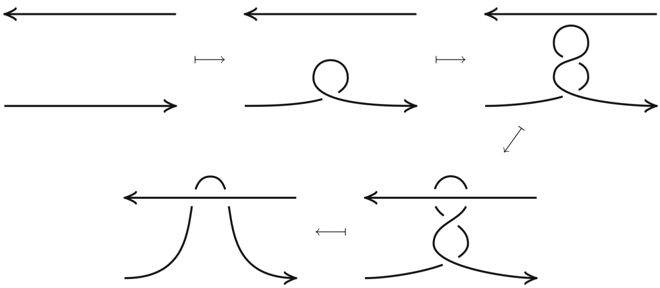
Show that $h(D) = h(g(D))$, and show that because the height is greater than zero, there is a bending, r. More importantly, show that r and g commute.

We thus see that $g(d) = r^{-1}gr(D)$. So then, we can surround g with a sequence of bendings and tightenings, so the transformation $D \to g(D)$ consists of a sequence of bendings and tightenings and one move of type (b), which we call g', such that g' is performed on a link of height zero, which we call D'. If all of the Seifert circles of D' are oriented counterclockwise, then D' is a 0-diagram (think about it), and g' is an Ω-move. If they are oriented clockwise, then g' can be expanded as a composition of an isotopy of S^2 which transforms D' into a 0-diagram, an Ω-move, and the inverse isotopy.

Moves of type (c)

I will show that moves of type (c) consist of a series of bendings and tightenings. First, if a nonbraidlike move $\Omega_2^{\pm 1}$ is performed on two distinct Seifert circles, this is by definition a bending. If it is performed on only one Seifert Circle, then it expands as two $\Omega_1\pm$ moves, a bending, and a tightening, as follows:

Expansion of $\Omega_2^{\pm 1}$



An expansion of $\Omega_2^{\pm 1}$

So to show that moves of type (c) are a sequence of either bendings and tightenings and isotopies on $\mathbb{R}^2$ or a sequence of Ω-moves on 0-diagrams, we only need to show that moves of type (a) are.

Moves of type (a)

Exercise (pretty dang hard)

Show that the move $\Omega_1^{\pm 1}$ expands as a sequence of bendings, tightenings, isotopies of S^2, or Ω-moves on 0-diagrams

The End

The following Lemma, along with Lemma 4, implies Lemma 3.

Lemma 5

Two 0-diagrams in $\mathbb{R}^2$ related by a sequence of bendings, tightenings, and isotopies in S^2 can be related by a sequence of Ω-moves

You'll just have to trust me on this one.

Conclusion

To sum up, we have that Lemma 5 and 4 imply that any two 0-diagrams representing isotopic oriented links in $\mathbb{R}^3$ can be related by a sequence of Ω-moves, which is Lemma 3. This implies Lemma 2, which is that two closed briad diagrams in an annulus can be related by a sequence of moves $(\Omega_2^{br})\pm1, (\Omega_3^{br})^{\pm1}, \tilde{M}2^{\pm1}, \tilde{M}3^{\pm1}$, and isotopies in the class of oriented link diagrams in that annulus. This implies Lemma 1, which says that two closed braids in the solid torus V representing isotopic oriented links in $\mathbb{R}^3$ can be related by a sequence of moves $\hat{M}2^{\pm1}, \hat{M}3^{\pm1}$, and isotopies in V. This, along with the black box, which states that braids with closures which are isotopic in V are conjugate, implies Markov's theorem.

Exercise (highly non-trivial)

Determine a simple way of determining if two braids are M-equivalent, and hence if two knots are isotopic.

Any Questions?